

Homework 8

This assignment is due by **11:59pm on Friday 11/8** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit (with the exception of row operations).

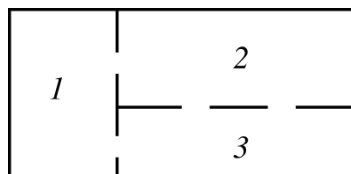
1. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

- (a) Find the eigenvalues of A . Show your work
- (b) Determine the eigenvalues of A^3 without computing A^3 . Explain your reasoning.
- (c) Determine the eigenvalues of A^T without computing A^T . Explain your reasoning.
- (d) Determine the eigenvalues of A^{-1} without computing A^{-1} . Explain your reasoning.

2. Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -2 & 1 & 5 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

- (a) Find the eigenvalues of A . Explain your reasoning.
 - (b) Find bases for the eigenspaces corresponding to the eigenvalues found in part (a). Show your work.
 - (c) What are the algebraic and geometric multiplicities of the eigenvalues? Explain your reasoning.
 - (d) Is A diagonalizable? Explain your reasoning.
3. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . Recall that the eigenspace corresponding to λ is $E_\lambda(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}$. In class, we saw that $E_\lambda(A) = N(A - \lambda I)$, which automatically implies it is a subspace of $\mathbb{R}^n$. Let's prove this a different way. Show that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$ using the definition of subspaces.
4. Show that A has an eigenvalue of 0 if and only if A is not invertible.
5. Let A be an $n \times n$ matrix with n eigenvalues (counted with multiplicity). Use the characteristic polynomial to show that the determinant of A is the product of the eigenvalues of A .

6. A scientist places a mouse in a box with 3 compartments, as shown below.



Notice that there are two doors between compartments 2 and 3, there is one door between compartments 1 and 2, and there is one door between compartments 1 and 3. The mouse has been trained to select a door at random and move through it whenever a bell is rung. For example, when the mouse is in compartment 2, there is a $1/3$ probability that it will move to compartment 1, a $2/3$ probability that it will move to compartment 3, and a zero probability that the it will remain in compartment 2. Let M be the matrix whose (i, j) th entry contains the probability that the mouse will move to compartment i given that it is in compartment j .

- Write down M and show that it is a regular Markov matrix.
- Find the steady state vector of M . Show your work.
- Suppose the mouse is placed in compartment i . Then at the start of the experiment, the initial probability vector is $\mathbf{x}_0 = \mathbf{e}_i$. This vector tells us that the probability that the mouse is in compartment i is 1 and the probability that the mouse is in one of the other compartments is 0. Consider the Markov chain $\mathbf{x}_k = M^k \mathbf{x}_0$. After the k th ring of the bell, the j th entry of $\mathbf{x}_k$ is the probability that the mouse is in the j th compartment (where $j = 1, 2$, or 3). Without computation, find $\lim_{k \rightarrow \infty} \mathbf{x}_k$. Explain your reasoning.
- Assuming that the bell is rung in regular intervals, what does the steady-state vector tell you about how much time the mouse spends in each compartment, in the long run?

7. This problem considers a Markov matrix that is not regular. Let $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

- Diagonalize M by hand; that is, find matrices P and D such that $M = PDP^{-1}$.
- Use part (a) to write down a formula for M^k . Deduce that M is not regular.
- Find all steady state vectors of M . Show your work.
- For the following initial state vectors, determine to which steady state vector (if any) the Markov chain $\mathbf{x}_k = M^k \mathbf{x}_0$ converges. Show your work

$$(i) \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (ii) \quad \mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} \quad (iii) \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}$$